

Bounding Run-Times of Local Adiabatic Algorithms

M. V. Panduranga Rao

Department of Computer Science and Automation
Indian Institute of Science
Bangalore
India.
pandurang@csa.iisc.ernet.in

Abstract. A common trick for designing faster quantum adiabatic algorithms is to apply the adiabaticity condition locally at every instant. However it is often difficult to determine the instantaneous gap between the lowest two eigenvalues, which is an essential ingredient in the adiabaticity condition. In this paper we present a simple linear algebraic technique for obtaining a lower bound on the instantaneous gap even in such a situation. As an illustration, we investigate the adiabatic unordered search of van Dam et al. [17] and Roland and Cerf [15] when the non-zero entries of the diagonal final Hamiltonian are perturbed by a polynomial (in $\log N$, where N is the length of the unordered list) amount. We use our technique to derive a bound on the running time of a local adiabatic schedule in terms of the minimum gap between the lowest two eigenvalues.

1 Introduction

Adiabatic Quantum Computation (AQC) has attracted a lot of interest in recent times. First introduced by Farhi et al. [11], this paradigm of computing makes use of the adiabatic theorem of quantum mechanics. Informally, the adiabatic theorem says that if a physical system is in the ground state of an initial Hamiltonian that evolves “slowly enough” to a final Hamiltonian, with a non-zero gap between the ground state and the first excited state of the Hamiltonian at all times, then the probability that the system ends up in the ground state of the final Hamiltonian approaches unity as the total time of evolution tends to infinity. This fact is used for solving computational problems as follows. To begin with, the system is in the ground state of a suitable Hamiltonian. This initial Hamiltonian is slowly evolved to a final Hamiltonian whose ground state represents the solution to the problem. If the running time required for a high probability of reaching the ground state of the final Hamiltonian is at most polynomial in the size of the input, we have an efficient AQC algorithm for the problem. The maximum rate at which the Hamiltonian can evolve at any instant without violating the adiabaticity condition depends inversely on square of the gap between the two instantaneous lowest eigenvalues. In many cases, it is difficult to estimate the gap at every instant of the evolution. Then, we strike a compromise by imposing a constant *global* delay schedule determined conservatively by the minimum gap over the entire interval. However, if we do have an estimate of this gap for the entire duration, we can apply the adiabaticity condition *locally* and speed up the rate of evolution wherever possible.

In the early days of AQC, efforts were largely focused on adiabatic optimization algorithms. Given a function $\phi : \{0, 1\}^n \rightarrow \mathbb{R}$, the problem is to find an $x \in \{0, 1\}^n$ that minimizes ϕ .

van Dam et al. [17] and Roland and Cerf [15] demonstrated the quantum nature of AQC by showing a quadratic speed-up for unordered search, matching Grover’s discrete algorithm [12].

However, any adiabatic quantum computer can be efficiently simulated by a standard quantum computer [17]. Applications of AQC techniques to small sized random instances of several NP complete problems like EXACT-COVER [8,9], CLIQUES [6] and SAT [8,9,11] have been explored with some success. Aharonov et al. [1] generalized this model by considering local¹ final Hamiltonians instead of only diagonal ones and showed that this generalized model can efficiently simulate any discrete quantum computation. Thus, in this general sense, AQC is equivalent to discrete quantum computation.

Adiabatic quantum computation is particularly interesting because of indications that it is more resilient to decoherence and implementation errors than the discrete model. While most schemes for implementing AQC oracles involve approximating them by a sequence of discrete unitary gates [3,17], robustness of potential implementations of the time-dependent Hamiltonians has also been studied. For example, Childs et al. [7] considered errors due to environmental decoherence and imperfect implementation in the latter approach. If the time dependent algorithm Hamiltonian is $H(t)$, they considered the actual Hamiltonian to be $H(t) + K(t)$, where $K(t)$ is an error Hamiltonian. In particular, $K(t)$ can be a perturbation in the final Hamiltonian. Through numerical simulations, they demonstrated robustness of AQC for small instances of combinatorial search problems against such errors. Åberg et al. [4,5] investigated robustness to decoherence in the instantaneous eigenvectors for local [5] and global adiabatic quantum search [4]. They showed that as long as Hamiltonian dynamics is present, asymptotic time complexity is preserved in both local and global cases. However, in case of pure decoherence, the time complexity of local search climbs to that of classical. For global evolution, it becomes costlier than classical: $N^{\frac{3}{2}}$, where N is the size of the list.

In this paper, we show how to use simple results from linear algebra to obtain bounds on the running time of algorithms that obey the adiabaticity condition locally. The technique is useful when the gap between the two lowest eigenvalues is not known at every instant of the evolution. As an illustration and running example, we investigate the behaviour of the eigenvalue spectrum when the final oracle Hamiltonian for the unordered search problem is perturbed in all non-zero elements by an amount at most polynomial in $\log N$.

In the unperturbed case, the gap between the lowest two eigenvalues is specified at every instant by a nice closed form expression [15,17]. This makes it rather easy to apply local adiabaticity. Perturbation deprives us of this facility. We show a work-around by lower-bounding the gap between the two lowest eigenvalue curves with straight lines whose slopes are obtained using the Wielandt-Hoffman theorem from the eigenvalue perturbation theory of symmetric matrices. The schedule can then be adjusted to satisfy the adiabaticity condition locally. The bound obtained by our method is commensurate with existing results—we obtain only a polynomial speed-up over the global algorithm [10,16,18]. Tighter bounds on eigenvalue perturbations will possibly yield Grover speed-up, indicating the resilience of the adiabatic algorithm to perturbations in the final

¹ A Hamiltonian is said to be local if it involves interactions between only a constant number of particles.

Hamiltonian. However, we believe that the techniques that we introduce in this paper can be applied in other AQC settings as well.

The paper is arranged as follows. The next section gives a brief discussion of the adiabatic quantum computing paradigm. Section 3 discusses the adiabatic search problem and its perturbed version. In section 4.1 we show that for the Hamiltonian in question, there exists a non-zero gap between the two lowest eigenvalues at all times. The proof largely follows that of Rao [14], simplified for the present case. However the minimum gap turns out to be exponentially small, and therefore a global schedule is of limited value. In section 4.2 we show our method to obtain a local schedule that provides a polynomial speed-up. Section 5 concludes the paper.

2 Preliminaries

In this section we give a brief overview of the adiabatic gap theorem and its application to quantum computing. For details the reader is referred to the text by Messiah [13]. Let $H(s)$, ($0 \leq s \leq 1$), be a time dependent single-parameter Hamiltonian for a system having an N dimensional Hilbert space. Let the eigenstates of $H(s)$ be given by $|l; s\rangle$ and the eigenvalues by $\lambda_l(s)$, with $\lambda_0(s) \leq \lambda_1(s) \leq \dots \leq \lambda_{N-1}(s)$ for $0 \leq s \leq 1$. Suppose we start with the initial state of the system $|\psi(0)\rangle$ as $|0; 0\rangle$ (the ground state of $H(0)$) and apply the Hamiltonian $H(s)$, $0 \leq s \leq 1$, to evolve it to $|\psi(1)\rangle$ at $s = 1$. Then the quantum adiabatic theorem states that for a “large enough” delay, the final state of the system $|\psi(1)\rangle$ will be arbitrarily close to the ground state $|0; 1\rangle$ of $H(1)$. Specifically, $|\langle 0; 1 | \psi(1) \rangle|^2 \rightarrow 1$ if the delay schedule $\tau(s)$ satisfies the adiabaticity condition at every s :

$$\tau(s) \gg \frac{\|\frac{d}{ds}H(s)\|_2}{g(s)^2}, \quad (1)$$

where $g(s)$ is the gap between the two lowest eigenvalues at s . In case $g(s)$ is difficult to determine for every s , which is the case with most Hamiltonians, we impose a more conservative *global* delay schedule for the entire duration T of the evolution:

$$T \gg \frac{\max_{0 \leq s \leq 1} \|\frac{d}{ds}H(s)\|_2}{g_{min}^2}, \quad (2)$$

where $g_{min} = \min_{0 \leq s \leq 1} (\lambda_1(s) - \lambda_0(s))$. However, if we do have an estimate of $g(s)$ for every s , we can use a varying delay schedule that satisfies the adiabaticity condition *locally* at every instant $0 \leq s \leq 1$. Then we can reduce the running time to

$$\int_{s=0}^1 \frac{\|\frac{d}{ds}H(s)\|_2 ds}{g(s)^2}. \quad (3)$$

If $H(s)$ is a polynomial-sized linear interpolation, $\|\frac{d}{ds}H(s)\|_2$ is a polynomial-sized quantity independent of s and the running time is of the order $\int_{s=0}^1 \frac{ds}{g(s)^2}$.

3 Perturbed Unordered Search

Consider a list of $N = 2^n$ elements, say bit-strings from $\{0, 1\}^n$. The unordered search problem may be stated as follows. Given a function $g : \{0, 1\}^n \rightarrow \{0, 1\}$ such that $g(u) = 0$ for a special element u and 1 for all others, find u . While any classical algorithm requires $O(N)$ queries to g , Grover's celebrated "discrete" quantum algorithm accomplishes the search in $O(\sqrt{N})$ queries only [12]. A similar speed-up was demonstrated by van Dam et al. [17] and Roland and Cerf [15] for this problem with AQC, which we discuss now. To begin with, note that the initial Hamiltonian $H(0)$ should be independent of the solution, with the restriction that it should not be diagonal in the computational basis [17]. Moreover, the ground state of the initial Hamiltonian should be a uniform superposition of all candidate solutions and easy to prepare. The following Hamiltonian satisfies the above conditions for searching an unordered list [17,15]:

$$H(0) = \sum_{z \in \{0,1\}^n \setminus 0^n} |\hat{z}\rangle\langle\hat{z}|, \quad (4)$$

where each $|\hat{z}\rangle$ is a basis vector in the 'Hadamard' basis given by

$$|\hat{z}\rangle = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n} |z\rangle$$

and the ground state is $\frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} |z\rangle$, which is easy to construct. The final Hamiltonian, then, is $\sum_{z \in \{0,1\}^n \setminus u} |z\rangle\langle z|$.

In this paper, we consider the case when the final Hamiltonian is perturbed in the non-zero entries. In other words, the final Hamiltonian is given by

$$H(1) = \sum_{z \in \{0,1\}^n} f(z) |z\rangle\langle z|,$$

where $\{|z\rangle\}$ form the "computational basis" of the Hilbert space of the system and $f : \{0, 1\}^n \rightarrow \mathbb{R}_{poly(n)}$ is a function that behaves as follows. $f(u) = 0$ for a special element $u \in \{0, 1\}^n$. For all other elements z , $f(z) > 0$. The problem is to minimize f , that is, to find u .

Given the initial and final Hamiltonians, define the interpolating Hamiltonian as

$$H(s) = (1 - s)H(0) + sH(1). \quad (5)$$

We start in the ground state of $H(0)$ and evolve to $H(1)$ slowly enough and end up in its ground state. Since $f(z)$ is bounded by a polynomial in n for all $z \in \{0, 1\}^n$, so is $\|\frac{d}{ds}H(s)\|_2$. The factor deciding the running time is therefore the denominator g_{min}^2 .

4 The Bounds

In what follows, we will assume $f(z) \neq f(z')$ for $z \neq z'$. The motivation for this assumption is two-fold: first, it makes for a neater presentation of our technique; secondly, given

the random nature of noise and implementational error, it is reasonable to assume that no two non-zero elements of the diagonal will be perturbed by the same amount. For the case when there do exist $z \neq z'$ such that $f(z) = f(z')$, the subsequent discussion requires only minor modification.

4.1 Global Evolution

By the nature of $H(s)$, the characteristic equation is independent of the permutations of the diagonal elements of the final Hamiltonian. Therefore, without loss of generality, we will follow the convention that $f(u) = 0 < f(z_1) < \dots < f(z_{N-1})$. First, we make sure that the gap between the two lowest eigenvalues is non-zero at all times. For that, we evaluate the characteristic equation of $H(s)$, in much the same way as in lemma 1 of [14], and [18].

Lemma 1. *The characteristic equation of $H(s)$ is*

$$c(\lambda) = (1 - s - \lambda) \left[\prod_{k=1}^{N-1} (1 - s + sf(z_k) - \lambda) - \frac{1-s}{N} \sum_{j=1}^{N-1} \frac{\prod_{k=1}^{N-1} (1-s+sf(z_k)-\lambda)}{1-s+sf(z_j)-\lambda} \right] - \frac{1-s}{N} \prod_{k=1}^{N-1} (1 - s + sf(z_k) - \lambda) = 0.$$

Proof. To evaluate the eigenvalue curves, we evaluate $|H(s) - \lambda I| = 0$. Subtracting the last column of this determinant from all other columns and using x_0 for $1 - s - \lambda$, x_1 for $1 - s + sf(z_1) - \lambda$ and so on, up to x_{N-1} for $1 - s + sf(z_{N-1}) - \lambda$, we have

$$\begin{vmatrix} x_0 & 0 & 0 & \dots & \dots & -\frac{(1-s)}{N} \\ 0 & x_1 & 0 & \dots & \dots & -\frac{(1-s)}{N} \\ \vdots & 0 & \ddots & \vdots & x_{\frac{N}{2}-1} & \vdots \\ -x_{N-1} & -x_{N-1} & \dots & \dots & -x_{N-1} & (x_{N-1} - \frac{(1-s)}{N}) \end{vmatrix}_{N \times N} = 0.$$

Expanding this determinant gives the required characteristic equation. \square

The following lemma is reminiscent of lemma 2 of [14], and [18]. We provide a somewhat simpler proof for the present case.

Lemma 2. *There exists exactly one root (i.e. an eigenvalue curve) of the characteristic equation in the intervals $(0, 1 - s)$, $(1 - s, 1 - s + sf(z_1))$, \dots , $(1 - s + sf(z_{N-2}), 1 - s + sf(z_{N-1}))$; for $0 < s < 1$.*

Proof. For any interval, let the $c_l(\lambda)$ and $c_u(\lambda)$ denote the value of the characteristic polynomial at the lower and upper boundary respectively.

We analyze the problem as three cases:

(1) The interval $(0, 1 - s)$:

In this case, $c_l(\lambda) = (1 - s) \left[\prod_{k=1}^{N-1} (1 - s + sf(z_k)) - \frac{1-s}{N} \sum_{j=1}^{N-1} \frac{\prod_{k=1}^{N-1} (1-s+sf(z_k))}{1-s+sf(z_j)} \right] - \frac{1-s}{N} \prod_{k=1}^{N-1} (1 - s + sf(z_k))$. This is a positive quantity for $s \in (0, 1)$, as can easily be

verified. Moreover, $c_u(\lambda) = -\frac{1-s}{N} \prod_{k=1}^{N-1} s f(z_k)$ is negative. Therefore, there exists a root in the interval $(0, 1-s)$ for $0 < s < 1$.

(2) The interval $(1-s, 1-s+sf(z_1))$:

Clearly, $c_l(\lambda)$ for this interval is same as $c_u(\lambda)$ of the previous case, which is negative.

But $c_u(\lambda) = -sf(z_1) \left[-\frac{1-s}{N} \prod_{k=2}^{N-1} s(f(z_k) - f(z_1)) \right]$ is positive. Thus, there exists at least one root in this interval also.

(3) The intervals $(1-s+sf(z_i), 1-s+sf(z_{i+1}))$, $1 \leq i \leq N-2$:

The values of the characteristic polynomial at the boundaries are respectively

$$c_l(\lambda) = -sf(z_i) \left[-\frac{1-s}{N} \prod_{k=1}^{i-1} s(f(z_k) - f(z_i)) \prod_{k=i+1}^{N-1} s(f(z_k) - f(z_i)) \right] \text{ and}$$

$$c_u(\lambda) = -sf(z_{i+1}) \left[-\frac{1-s}{N} \prod_{k=1}^i s(f(z_k) - f(z_{i+1})) \prod_{k=i+2}^{N-1} s(f(z_k) - f(z_{i+1})) \right].$$

Notice that in either case, every element in the first product is negative and that in the second is positive. This is because $f(z_k) < f(z_i)$ for $1 \leq k \leq i-1$ and $f(z_k) > f(z_i)$ for $i+1 \leq k \leq N-1$. Therefore, the over-all sign is decided by the number of elements in the first product only. But the number of such elements in $c_l(\lambda)$ and $c_u(\lambda)$ differ by one. Thus $c_l(\lambda)$ and $c_u(\lambda)$ differ in sign. Therefore, there exists a root in each of these intervals.

Hence, there is a root in each interval. Given that (i) there are N open intervals bounded by N straight line eigenvalue curves, (ii) there is at least one root in each interval and (iii) there are N roots in all, the lemma follows. \square

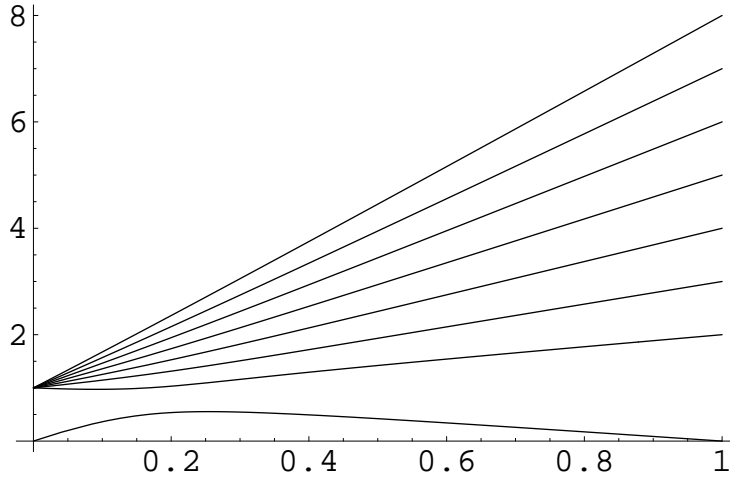


Fig. 1. Example Eigenvalue Curves

By virtue of the above lemma, we can speak of one curve being “above” another. We label the curves as $\lambda_0(s), \lambda_1(s), \dots, \lambda_{N-1}(s)$, starting from below. Thus, $\lambda_0(s)$ lies between the lines $\lambda(s) = 0$ and $\lambda_1(s)$. See figure 1 for example.

As a consequence of the above lemma, we are guaranteed a non-zero gap between $\lambda_0(s)$ and $\lambda_1(s)$ for all $s \in [0, 1]$.

What about the minimum gap between $\lambda_0(s)$ and $\lambda_1(s)$? It turns out that this gap is exponentially small [18]. Making use of the properties of the present problem, we arrive at this conclusion in a different manner.

We first discuss the case when $f(z_1)$ is large, say greater than 1. Consider the characteristic polynomial $c(\lambda) = (1-s-\lambda) \left[\prod_{k=1}^{N-1} (1-s+sf(z_k)-\lambda) - \frac{1-s}{N} \sum_{j=1}^{N-1} \frac{\prod_{k=1}^{N-1} (1-s+sf(z_k)-\lambda)}{1-s+sf(z_j)-\lambda} \right] - \frac{1-s}{N} \prod_{k=1}^{N-1} (1-s+sf(z_k)-\lambda)$.

By lemma 2, the line $\lambda(s) = 1-s$ separates λ_0 and λ_1 . Thus, if

(i) there exists an s_0 where the above polynomial is divisible by two “ $(1-s_0-\lambda)$ factors” and

(ii) all other curves ($\lambda_2(s)$ through $\lambda_{N-1}(s)$) are at least an inverse polynomial distance above the point $(s_0, 1-s_0)$,

it would imply that $\lambda_0(s_0)$ and $\lambda_1(s_0)$ are exponentially close to the line $\lambda(s) = 1-s$, and in turn, to each other.

Clearly, points very close to $s = 0$ are not candidates, as there are too many curves in the vicinity. Points close to $s = 1$ are also ruled out, as there is only one $(1-s-\lambda)$ factor, corresponding to the curve $\lambda_0(s)$: $\lambda_1(s)$ is $f(z_1)$ units above.

Consider a candidate point $s_0 = \frac{1}{p(n)}$ where $p(n)$ is a polynomial in n such that $p(n) > \max_k(f(z_k))$. At $s_0 = \frac{1}{p(n)}$, $\lambda_2 > 1 + \frac{f(z_1)-1}{p(n)}$. Thus, by lemma 2, all curves above $\lambda_2(s)$ are at least an inverse polynomial distance above $1-s_0 = 1 - \frac{1}{p(n)}$.

Notice that the polynomial can be divided by one $(1-s-\lambda)$ factor at an s such that $\frac{1-s}{N} \prod_{k=1}^{N-1} (1-s+sf(z_k)-\lambda) \rightarrow 0$. Let us test the candidate point s_0 . Substituting $1-s$ for λ and $\frac{1}{p(n)}$ for s , we get $\frac{p(n)-1}{Np(n)} \prod_{k=1}^{N-1} (\frac{f(z_k)}{p(n)})$, which indeed tends to zero. Therefore, the existence of one $(1-s-\lambda)$ factor at s_0 has been established. Let us now see if there exists another. On dividing the polynomial by $(1-s_0-\lambda)$ at $s_0 = \frac{1}{p(n)}$, we obtain

$$c(\lambda) = \left[\prod_{k=1}^{N-1} \frac{f(z_k)}{p(n)} - \frac{p(n)-1}{Np(n)} \sum_{j=1}^{N-1} \frac{\prod_{k=1}^{N-1} \frac{f(z_k)}{p(n)}}{\frac{f(z_j)}{p(n)}} \right], \text{ which tends to zero.}$$

Suppose now that $f(z_1)$ is “small”. If $f(z_1)$ is a constant independent of n or even $O(\frac{1}{\text{poly}(n)})$, the argument given above holds exactly. However, if $f(z_1)$ is greater than zero by only an exponentially small quantity, the two $(1-s-\lambda)$ factors that we obtained in the above discussion could correspond to $\lambda_2(s)$, thus leaving the possibility of $\lambda_0(s)$ being at a larger distance below $\lambda(s) = 1-s$. But this is not the case. Since by assumption $\lambda_2(s)$ is exponentially close to the line $\lambda(s) = 1-s$, it can be factored out from the polynomial, leaving two factors that correspond to $\lambda_0(s)$ and $\lambda_1(s)$.

Thus, it turns out that the minimum gap is too small to yield any significant speed-up for a constant rate global adiabatic algorithm.

4.2 Local Evolution

We will now obtain an upper bound on $\int_{s=0}^1 \frac{d(s)}{g(s)^2}$ where $g(s) = \lambda_1(s) - \lambda_0(s)$, for local evolution. To that end, the following result from the perturbation theory of real symmetric matrices will be useful.

Theorem 1. *Wielandt-Hoffman Theorem (WHT): Let A and E be real symmetric matrices and let $\hat{A} = A + E$. Let the eigenvalues of A be $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}$ and those of \hat{A} be $\hat{\lambda}_0 \leq \hat{\lambda}_1 \leq \dots \leq \hat{\lambda}_{N-1}$.*

Then

$$\sum_{j=0}^{N-1} (\lambda_j - \hat{\lambda}_j)^2 \leq \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |E_{ij}|^2. \quad (6)$$

□

Let E of WHT be the ‘perturbation matrix’ $H(s + ds) - H(s)$. This provides us with the r.h.s. of (6):

$$\left(\frac{N-1}{N}\right)^2 (ds)^2 + \frac{N-1}{N} (ds)^2 + \sum_{k=1}^{N-1} \left(f(z_k) - \frac{N-1}{N}\right)^2 (ds)^2.$$

For the l.h.s., note that $\lambda_1(s) \dots \lambda_{N-1}(s)$ are $N-1$ non-crossing curves packed inside a polynomially bounded gap and each curve is bounded by two straight lines by lemma 2.

Thus, there are at most a polynomial $q(n)$ number of the gaps $f(z_i) - f(z_{i-1})$ that are at least inverse polynomially wide. All other curves are packed between the straight lines $\lambda = (f(z_i) - 1)s + 1$ and $\lambda = (f(z_{i-1}) - 1)s + 1$ where $f(z_i) - f(z_{i-1}) = O(\frac{n^c}{N})$ for a constant c . Therefore the slopes of these curves can be approximated by the slopes of one of the enclosing lines. Hence, the l.h.s. is

$$d\lambda_0^2 + d\lambda_{i_1}^2 + \dots + d\lambda_{i_{q(n)}}^2 + \sum_{k \in \{1, \dots, N-1\} \setminus \{i_1, \dots, i_{q(n)}\}} (f(z_k) - 1)^2 (ds)^2$$

Substituting in 6, we get an upper bound on $d\lambda_0^2 + d\lambda_{i_1}^2 + \dots + d\lambda_{i_{q(n)}}^2$. In particular, this is also an upper bound on $\frac{d\lambda_0}{ds}$ and $\frac{d\lambda_1}{ds}$.² This can be used to obtain an upper bound on the total time required, as we will see soon.

We illustrate the technique for the case when $q(n) = 1$ and $f(z_1) \geq 1$.

Lemma 3. *If $q(n) = 1$ and $f(z_1) \geq 1$,*

$$\left(\frac{d\lambda_0(s)}{ds}\right)^2 + \left(\frac{d\lambda_1(s)}{ds}\right)^2 \leq (f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N} \sum_{k=1}^{N-2} f(z_k). \quad (7)$$

² A similar bound can be obtained using a general result of Ambainis and Regev ([2], Lemma 4.1). Nevertheless, we presented a different way to demonstrate the possibility of problem specific approaches which can yield tighter bounds than the general one.

Proof. By the preceding argument, the l.h.s. of 6 is given by

$$\sum_{k=1}^{N-2} (f(z_k) - 1)^2 (ds)^2 + (d\lambda_0(s))^2 + (d\lambda_1(s))^2. \text{ Substituting in equation 6, we get}$$

$$\sum_{k=1}^{N-2} (f(z_k) - 1)^2 (ds)^2 + (d\lambda_0(s))^2 + (d\lambda_1(s))^2$$

$$\leq \left(\frac{N-1}{N}\right)^2 (ds)^2 + \frac{N-1}{N} (ds)^2 + \sum_{k=1}^{N-1} \left(f(z_k) - \frac{N-1}{N}\right)^2 (ds)^2.$$

Rearranging some terms we get,

$$\left(\frac{d\lambda_0}{ds}\right)^2 + \left(\frac{d\lambda_1}{ds}\right)^2 \leq \left(\frac{N-1}{N}\right)^2 + \frac{N-1}{N} + (f(z_{N-1}) - \frac{N-1}{N})^2$$

$$+ \sum_{k=1}^{N-2} \left((f(z_k) - \frac{N-1}{N})^2 - (f(z_k)s - \frac{N-1}{N})^2\right)^2.$$

Or,

$$\left(\frac{d\lambda_0}{ds}\right)^2 + \left(\frac{d\lambda_1}{ds}\right)^2 \leq \left(\frac{N-1}{N}\right)^2 + \frac{N-1}{N} + (f(z_{N-1})s - \frac{N-1}{N})^2 + \frac{1}{N} \sum_{k=1}^{N-2} (2n + 2f(z_k) - \frac{2N-1}{N}).$$

Simplifying and ignoring small terms, we get

$$\left(\frac{d\lambda_0}{ds}\right)^2 + \left(\frac{d\lambda_1}{ds}\right)^2 \leq (f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N} \sum_{k=1}^{N-2} f(z_k).$$

□

The lemma implies that $|\frac{d\lambda_0}{ds}|, |\frac{d\lambda_1}{ds}| \leq \sqrt{(f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N} \sum_{k=1}^{N-2} f(z_k)}$.

Using this, we will estimate $g(s) = \lambda_1(s) - \lambda_0(s)$. We conservatively approximate $\lambda_0(s)$ and $\lambda_1(s)$ by straight lines to get a lower bound on $g(s) = \lambda_1(s) - \lambda_0(s)$.

We divide the interval $[0, 1]$ into three parts $[0, a]$, $[a, b]$ and $[b, 1]$, corresponding to the parts when λ_0 approaches $\lambda = 1 - s$, when both λ_0 and λ_1 are close to $\lambda = 1 - s$, and when $\lambda_1(s)$ rises away from $\lambda = 1 - s$ respectively.

Consider the first interval. Suppose $l_0^1(s)$ and $l_1^1(s)$ are lines such that $l_0^1(s) \geq \lambda_0(s)$ and $l_1^1(s) \leq \lambda_1(s)$. Then the gap between these lines is a lower bound on $\lambda_1(s) - \lambda_0(s)$ in the interval $(0, a)$.

Denote $\sqrt{(f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N} \sum_{k=1}^{N-2} f(z_k)} - 1$ by m . We take $l_0^1(s) = ms$ and $l_1^1(s) = 1 - s$. Thus, the gap in the first interval is given by $g_1(s) \geq 1 - s - ms$. Similarly, for the third interval, we choose $l_0^3(s) = 1 - s$ and $l_1^3(s) = ms - m + f(z_1)$. Therefore, the gap $g_3(s)$ in this interval is greater than $ms - m + f(z_1) - (1 - s)$. We will conservatively take the gap in the entire second interval to be g_{min} . See figure 2 for a rough sketch.

Let a be specified by a point on $l_0^1(s) = ms$ that lies vertically below the line $\lambda = 1 - s$ by a distance g_{min} . Thus, $a = \frac{1 - g_{min}}{m + 1}$. Similarly, b is specified by a point on the line $l_1^3(s) = ms - m + f(z_1)$ that is above $\lambda = 1 - s$ by g_{min} . Then, $b = \frac{m + 1 - f(z_1) + g_{min}}{m + 1}$.

For the gap $g_2(s)$, recall that one of λ_0 and λ_1 is away from the line $1 - s$ by a margin of g_{min} . Since this gives the deviation of only one of λ_0 and λ_1 from $1 - s$, this is a conservative estimate of $g_2(s)$.

Therefore the total delay factor is given by

$$T \gg \int_0^a \frac{ds}{g_1(s)^2} + \int_a^b \frac{ds}{g_2(s)^2} + \int_b^1 \frac{ds}{g_3(s)^2}$$

$$\leq \int_0^a \frac{ds}{(1 - s(1 + m))^2} + \frac{1}{g_{min}^2} \int_a^b ds + \int_b^1 \frac{ds}{((m + 1)(s - 1) + f(z_1))^2}$$

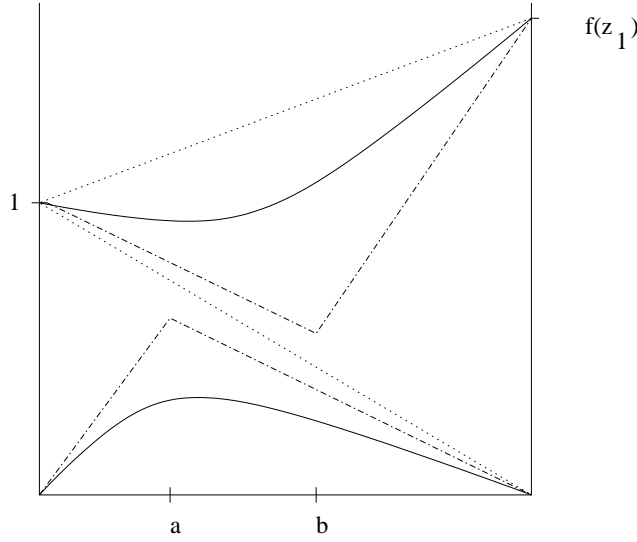


Fig. 2. Bold lines show actual eigenvalue curves and the dashed lines show lines that are assumed to bound the gap.

$$\begin{aligned}
&= \frac{1}{m+1} \left(\frac{1}{g_{\min}} - 1 \right) + \frac{m + 2g_{\min} - f(z_1)}{g_{\min}^2(m+1)} + \frac{1}{m+1} \left(\frac{1}{g_{\min}} - \frac{1}{f(z_1)} \right) \\
&\approx \frac{m - f(z_1)}{(m+1)g_{\min}^2}.
\end{aligned}$$

Let us summarize:

Theorem 2. *Let the non-zero elements of the final Hamiltonian of the local adiabatic search algorithm be $f(z_1) \neq f(z_2) \neq f(z_{N-1})$, where each $f(z_i)$ is of size at most $O(\log N)$. Then, the time taken to evolve to the solution state of the final Hamiltonian is $O(D \frac{m-f(z_1)}{(m+1)g_{\min}^2})$, where $m = \sqrt{(f(z_{N-1}) - 1)^2 + 2n + \frac{2}{N} \sum_{k=1}^{N-2} f(z_k) - 1}$, $g_{\min} = \min_{0 \leq s \leq 1} (\lambda_1(s) - \lambda_0(s))$, and $D = \|\frac{d}{ds} H(s)\|_2$.*

To obtain tighter upper bounds on the running time, we would require the interval $b - a$ to be shorter: short enough to balance the denominator g_{\min}^2 in the second term. This in turn requires tighter bounds on $\frac{d\lambda_0}{ds}$ and $\frac{d\lambda_1}{ds}$.

5 Conclusions

We introduced a technique for obtaining upper bounds on the running time of adiabatic quantum algorithms if the eigenvalue spectrum behaves in certain ways. We used this technique to investigate the robustness of the adiabatic quantum algorithm for unordered search when the final Hamiltonian is perturbed in the non-zero entries. Interesting open problems include tightening of the bounds, and application of our technique to other quantum adiabatic algorithms.

References

1. D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev. Adiabatic quantum computation is equivalent to standard quantum computation. In *Annual IEEE Symposium on Foundations of Computer Science*, pages 42–51, 2004.
2. Andris Ambainis and Oded Regev. An elementary proof of the quantum adiabatic theorem, 2004.
3. M. Andrecut and M. K. Ali. Adiabatic quantum oracles. *Journal of Physics A: Mathematical and General*, 37:L421–L427, 2004.
4. J. Åberg, D. Kult, and E. Sjöqvist. Quantum adiabatic search with decoherence in the instantaneous energy eigenbasis. *Physical Review A*, 72(042317), 2005.
5. J. Åberg, D. Kult, and E. Sjöqvist. Robustness of the adiabatic quantum search. *Physical Review A*, 71(060312(R)), 2005.
6. A. M. Childs, E. Farhi, J. Goldstone, and S. Gutmann. Finding cliques by quantum adiabatic evolution. *Quantum Information and Computation*, 2(3):181–191, 2002.
7. A. M. Childs, E. Farhi, and J. Preskill. Robustness of adiabatic quantum computation. *Physical Review A*, 65(012322), 2002.
8. E. Farhi, J. Goldstone, and S. Gutmann. A numerical study of the performance of a quantum adiabatic evolution algorithm for satisfiability. *quant-ph/0007071*, 2000.
9. E. Farhi, J. Goldstone, S. Gutmann, J. Lapan, A. Lundgren, and D. Preda. A quantum adiabatic evolution algorithm applied to random instances of an np-complete problem. *Science*, 292(5516):472–476, 2001.
10. E. Farhi, J. Goldstone, S. Gutmann, and D. Nagaj. How to make the quantum adiabatic algorithm fail. *quant-ph/0512159*, 2005.
11. E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser. Quantum computation by adiabatic evolution. *arXiv:quant-ph/0001106*, 2002.
12. L. Grover. Quantum mechanics helps in searching for a needle in a haystack. *Physical Review Letters*, 79(2):325–328, 1997.
13. A. Messiah. *Quantum Mechanics*. John Wiley and Sons, New York, 1958.
14. M. V. Panduranga Rao. Solving a hidden subgroup problem using the adiabatic quantum computing paradigm. *Physical Review A*, 67:052306, 2003.
15. J. Roland and N. Cerf. Quantum search by local adiabatic evolution. *Physical Review A*, 65(042308), 2002.
16. G. Schaller, S. Mostame, and Ralf Schützhold. General error estimate for adiabatic quantum computing. *Physical Review A*, 73:062307, 2006.
17. W. van Dam, M. Mosca, and U. V. Vazirani. How powerful is adiabatic quantum computation?. In *Annual IEEE Symposium on Foundations of Computer Science*, pages 279–287, 2001.
18. M. Znidaric and M. Horvat. Exponential complexity of an adiabatic algorithm for an np-complete problem. *Physical Review A*, 73:022329, 2006.